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# NATURAL STRESS TENSORS

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*The notion of natural stress tensors is introduced, which are tensors obtained from Cauchy and Kirchhoff stress tensors by operations of mapping, half-mapping, and mixed mapping of the actual configuration onto the reference one, and also onto two intermediate configurations. A complete class of natural stress tensors is obtained and analyzed.*

**Introduction.** The operation of tensor mapping [1] (bringing one tensor in conformity to another [2]) from one configuration onto another is used in mechanics of continuous media. For second-rank tensors, this operation is as follows. At a certain material point, the tensor is defined by component decompositions into basis dyads in a chosen configuration. The mapped tensor is obtained by replacing these dyads by basis dyads at the same material point from another configuration. Since the tensor components remain unchanged, a set of different (mapped) tensors is obtained. By analogy, we introduce an operation of half-mapping, where only one basis vector rather than the whole basis dyad from the other configuration is replaced. We also introduce an operation of half-mapping, where one basis vector is replaced in the dyad by the corresponding vector from one chosen configuration, and the other vector is replaced by a vector from another configuration.

We consider the following configurations [3]: reference configuration, actual (current or deformed) configuration, current back-rotated configuration, and rotated reference configuration. Mechanics of continuous media employ the Cauchy stress tensor (true stress tensor) $s$ [2, 3]. Sometimes, it is more convenient to use the Kirchhoff stress tensor $\tau$ [3] (weight stress tensor [2]), which differs from the true stress tensor by a scalar factor $J$. We introduce the notion of natural stress tensors, which are tensors that can be obtained from the Cauchy and Kirchhoff stress tensors by operations of mapping, half-mapping, and mixed mapping of the image from the actual configuration onto the reference configuration, current back-rotated configuration, and rotated reference configuration. The essence of the term "natural" is as follows. The Cauchy and Kirchhoff stress tensors characterize the elementary force related to elementary areas in actual and reference configurations, respectively. For tensors obtained by operations of mapping of the Cauchy and Kirchhoff stress tensors, there exist basis dyads, where the components of the mapped vectors are equal to the components of tensors generating them with their mechanical meaning preserved. In a certain sense, all natural tensors are "equivalent" to the Cauchy or Kirchhoff stress tensors.

The objective of the present work is to determine the complete class of natural stress tensors and to identify, where possible, the tensors obtained with the known tensors.

**Kinematics of Deformation.** We consider the motion of a body $B$ in the Euclidian three-dimensional space, where a Cartesian coordinate system with orthonormal basis vectors $\boldsymbol{k}_i$ is introduced (hereinafter, the subscripts $i$ and $j$ run over the values of 1, 2, and 3). For generality, we consider a curvilinear coordinate system $\Theta^i$, which is the reference system, in addition to the Cartesian coordinate system. Let $P$ be a certain material point of the body $B$. Following [4], we call the material point with its infinitesimal vicinity the material particle. Let $\boldsymbol{X}$ and $\boldsymbol{x}$ be the radius-vectors of the point $P$ in the reference and current configurations, respectively. We assume that the transformation of the reference configuration into the actual one is described by the law of motion: by a continuous vector function with

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) \ : \ \boldsymbol{x}(\boldsymbol{X}, t_0) = \boldsymbol{X}$$

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under the condition

$$0 < J \equiv \det F < \infty \quad \forall\, t > t_0,$$

where $t$ is a monotonically increasing deformation parameter (time), $t_0$ is the value of the parameter $t$ corresponding to the reference configuration, and $F$ is the deformation-gradient tensor

$$F \equiv \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial x_i}{\partial X_j}\, \boldsymbol{k}_i \otimes \boldsymbol{k}_j \quad (\boldsymbol{X} = X_i \boldsymbol{k}_i,\ \boldsymbol{x} = x_i \boldsymbol{k}_i).$$

Hereinafter, the sign "$\otimes$" denotes the dyad product of the basis vectors; summation is performed over repeated indices. Polar decomposition of this tensor leads the equalities ($\det F > 0$)

$$F = R \cdot U = V \cdot R \quad (U^{\mathrm{t}} = U,\ V^{\mathrm{t}} = V,\ R \cdot R^{\mathrm{t}} = g,\ \det R = 1), \tag{1}$$

where $g$ is the metric (unit) tensor, $U$ and $V$ are the symmetric, positively defined (all principal values are positive) right and left stretch tensors), and $R$ is the proper orthogonal rotation tensor; the dot indicates the scalar (internal) product of tensors, and the superscript "t" implies the operation of tensor transposition. From Eq. (1), we obtain $U = R^{\mathrm{t}} \cdot V \cdot R$ and $V = R \cdot U \cdot R^{\mathrm{t}}$, i.e., the tensor $U$ may be obtained from the tensor $V$ by the operation of back rotation; vice versa, the tensor $V$ can be obtained from the tensor $U$ by the operation of rotation.

In transformation of coordinates corresponding to the rigid motion of the body, we have

$$\boldsymbol{x}^*(\boldsymbol{X}, t) \equiv Q(t) \cdot \boldsymbol{x}(\boldsymbol{X}, t) + \boldsymbol{c}(t), \tag{2}$$

where $Q(t)$ is the proper orthogonal tensor ($Q \cdot Q^{\mathrm{t}} = g$ and $\det Q = 1$) and $\boldsymbol{c}(t)$ is a vector; the tensors $U$ and $V$ are transformed as follows:

$$U^* = U, \quad V^* = Q \cdot V \cdot Q^{\mathrm{t}}. \tag{3}$$

We call the tensors that satisfy, in transformation (2), the first and second relations in (3) invariant and indifferent tensors, respectively. Invariant and indifferent tensors form the class of objective tensors [3]. Note that the tensor $U$ is invariant and the tensor $V$ is indifferent, i.e., they are objective, whereas the tensors $F$ and $R$ are not objective.

We define the Lagrangian coordinate system $\hat{\Theta}^i$ as follows: we assume that $\hat{\Theta}^i = \Theta^i$ in the reference configuration for $t = t_0$; at any other time $t > t_0$, the Lagrangian coordinates of a fixed material point have the same values of $\hat{\Theta}^i$. We determine the material reference and current covariant basis vectors as

$$\check{\boldsymbol{e}}_i(\hat{\Theta}^k) \equiv \frac{\partial \boldsymbol{X}}{\partial \hat{\Theta}^i} = \frac{\partial X_j}{\partial \hat{\Theta}^i}\, \boldsymbol{k}_j, \quad \hat{\boldsymbol{e}}_i(\hat{\Theta}^k, t) \equiv \frac{\partial \boldsymbol{x}}{\partial \hat{\Theta}^i} = \frac{\partial x_j}{\partial \hat{\Theta}^i}\, \boldsymbol{k}_j : \quad \hat{\boldsymbol{e}}_i(\hat{\Theta}^k, t_0) = \check{\boldsymbol{e}}_i(\hat{\Theta}^k).$$

In accordance with the formulas of tensor analysis, we also introduce covariant basis vectors $\check{\boldsymbol{e}}^i$ and $\hat{\boldsymbol{e}}^i$. The covariant and contravariant basis vectors in the reference and current configurations are related by the following transformation formulas [3]:

$$\begin{aligned} \hat{\boldsymbol{e}}_i &= F \cdot \check{\boldsymbol{e}}_i = \check{\boldsymbol{e}}_i \cdot F^{\mathrm{t}}, & \hat{\boldsymbol{e}}^i &= F^{-\mathrm{t}} \cdot \check{\boldsymbol{e}}^i = \check{\boldsymbol{e}}^i \cdot F^{-1}, \\ \check{\boldsymbol{e}}_i &= F^{-1} \cdot \hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_i \cdot F^{-\mathrm{t}}, & \check{\boldsymbol{e}}^i &= F^{\mathrm{t}} \cdot \hat{\boldsymbol{e}}^i = \hat{\boldsymbol{e}}^i \cdot F. \end{aligned} \tag{4}$$

Hereinafter $F^{-\mathrm{t}} \equiv (F^{-1})^{\mathrm{t}} = (F^{\mathrm{t}})^{-1}$.

We denote the reference and actual configurations introduced previously as $\check{B}$ and $\hat{B}$, respectively, and introduce two intermediate configurations [3].

1. The current back-rotated configuration $\bar{B}$ is obtained from the configuration $\hat{B}$ by the operation of back rotation. The basis vectors of the material current basis ($\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}^i$) and the material current back-rotated basis $\bar{\boldsymbol{e}}_i$ and $\bar{\boldsymbol{e}}^i$ are related via the transformation formulas

$$\begin{aligned} \bar{\boldsymbol{e}}_i &\equiv R^{\mathrm{t}} \cdot \hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_i \cdot R, & \bar{\boldsymbol{e}}^i &\equiv R^{\mathrm{t}} \cdot \hat{\boldsymbol{e}}^i = \hat{\boldsymbol{e}}^i \cdot R, \\ \hat{\boldsymbol{e}}_i &= R \cdot \bar{\boldsymbol{e}}_i = \bar{\boldsymbol{e}}_i \cdot R^{\mathrm{t}}, & \hat{\boldsymbol{e}}^i &= R \cdot \bar{\boldsymbol{e}}^i = \bar{\boldsymbol{e}}^i \cdot R^{\mathrm{t}}. \end{aligned} \tag{5}$$

2. The rotated reference configuration $\tilde{B}$ is obtained from the configuration $\check{B}$ by the operation of rotation. The basis vectors of the material reference basis $\check{\boldsymbol{e}}_i$, $\check{\boldsymbol{e}}^i$ and the rotated material reference basis $\tilde{\boldsymbol{e}}_i$, $\tilde{\boldsymbol{e}}^i$ are transformed as follows:

$$\begin{aligned} \tilde{\boldsymbol{e}}_i &\equiv R \cdot \check{\boldsymbol{e}}_i = \check{\boldsymbol{e}}_i \cdot R^{\mathrm{t}}, & \tilde{\boldsymbol{e}}^i &\equiv R \cdot \check{\boldsymbol{e}}^i = \check{\boldsymbol{e}}^i \cdot R^{\mathrm{t}}, \\ \check{\boldsymbol{e}}_i &= R^{\mathrm{t}} \cdot \tilde{\boldsymbol{e}}_i = \tilde{\boldsymbol{e}}_i \cdot R, & \check{\boldsymbol{e}}^i &= R^{\mathrm{t}} \cdot \tilde{\boldsymbol{e}}^i = \tilde{\boldsymbol{e}}^i \cdot R. \end{aligned} \tag{6}$$

These configurations are determined only by local transformations for each material particle of the body. In the general case, in contrast to the reference and actual configurations, they do not form real configurations of the deformed body.

Equations (1) and (4)–(6) yield the following transformation formulas for the basis vectors [3]:

$$
\begin{aligned}
&\check{\boldsymbol{e}}_i = U^{-1}\cdot\bar{\boldsymbol{e}}_i = \bar{\boldsymbol{e}}_i\cdot U^{-1}, && \check{\boldsymbol{e}}^i = U\cdot\bar{\boldsymbol{e}}^i = \bar{\boldsymbol{e}}^i\cdot U,\\
&\bar{\boldsymbol{e}}_i = U\cdot\check{\boldsymbol{e}}_i = \check{\boldsymbol{e}}_i\cdot U, && \bar{\boldsymbol{e}}^i = U^{-1}\cdot\check{\boldsymbol{e}}^i = \check{\boldsymbol{e}}^i\cdot U^{-1},\\
&\hat{\boldsymbol{e}}_i = V\cdot\tilde{\boldsymbol{e}}_i = \tilde{\boldsymbol{e}}_i\cdot V, && \hat{\boldsymbol{e}}^i = V^{-1}\cdot\tilde{\boldsymbol{e}}^i = \tilde{\boldsymbol{e}}^i\cdot V^{-1},\\
&\tilde{\boldsymbol{e}}_i = V^{-1}\cdot\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_i\cdot V^{-1}, && \tilde{\boldsymbol{e}}^i = V\cdot\hat{\boldsymbol{e}}^i = \hat{\boldsymbol{e}}^i\cdot V.
\end{aligned}
\tag{7}
$$

**Class of Natural Stress Tensors.** Let $s$ be an indifferent symmetric Cauchy stress tensor (true stress tensor) that characterizes the stress state of the material particle. We also introduce an indifferent symmetric Kirchhoff stress tensor

$$\tau \equiv Js. \tag{8}$$

We consider the following representations of this tensor:

$$\tau = \hat{\tau}^{ij}\,\hat{\boldsymbol{e}}_i\otimes\hat{\boldsymbol{e}}_j = \hat{\tau}_{ij}\,\hat{\boldsymbol{e}}^i\otimes\hat{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\,\hat{\boldsymbol{e}}^i\otimes\hat{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\,\hat{\boldsymbol{e}}_i\otimes\hat{\boldsymbol{e}}^j.$$

Using the operations of mapping the tensor $\tau$ from the actual $\hat{B}$ onto the reference configuration $\check{B}$, we obtain four stress tensors:

$$
\begin{aligned}
&S_1 \equiv \hat{\tau}^{ij}\,\check{\boldsymbol{e}}_i\otimes\check{\boldsymbol{e}}_j &&\Leftrightarrow\quad S_1 \equiv F^{-1}\cdot\tau\cdot F^{-\mathrm{t}},\\
&S_2 \equiv \hat{\tau}_{ij}\,\check{\boldsymbol{e}}^i\otimes\check{\boldsymbol{e}}^j &&\Leftrightarrow\quad S_2 \equiv F^{\mathrm{t}}\cdot\tau\cdot F,\\
&S_3 \equiv \hat{\tau}_i{}^j\,\check{\boldsymbol{e}}^i\otimes\check{\boldsymbol{e}}_j &&\Leftrightarrow\quad S_3 \equiv F^{\mathrm{t}}\cdot\tau\cdot F^{-\mathrm{t}},\\
&S_4 \equiv \hat{\tau}^i{}_j\,\check{\boldsymbol{e}}_i\otimes\check{\boldsymbol{e}}^j &&\Leftrightarrow\quad S_4 \equiv F^{-1}\cdot\tau\cdot F.
\end{aligned}
\tag{9}
$$

Using the operations of mixed mapping of the tensor $\tau$ from the actual configuration $\hat{B}$ onto the configurations $\check{B}$ and $\bar{B}$, we obtain four more stress tensors:

$$
\begin{aligned}
&S_5 \equiv \hat{\tau}^{ij}\,\bar{\boldsymbol{e}}_i\otimes\check{\boldsymbol{e}}_j = \hat{\tau}_i{}^j\,\bar{\boldsymbol{e}}^i\otimes\check{\boldsymbol{e}}_j &&\Leftrightarrow\quad S_5 \equiv R^{\mathrm{t}}\cdot\tau\cdot F^{-\mathrm{t}},\\
&S_6 \equiv \hat{\tau}^{ij}\,\check{\boldsymbol{e}}_i\otimes\bar{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\,\check{\boldsymbol{e}}_i\otimes\bar{\boldsymbol{e}}^j &&\Leftrightarrow\quad S_6 \equiv F^{-1}\cdot\tau\cdot R,\\
&S_7 \equiv \hat{\tau}_{ij}\,\check{\boldsymbol{e}}^i\otimes\bar{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\,\check{\boldsymbol{e}}^i\otimes\bar{\boldsymbol{e}}_j &&\Leftrightarrow\quad S_7 \equiv F^{\mathrm{t}}\cdot\tau\cdot R,\\
&S_8 \equiv \hat{\tau}_{ij}\,\bar{\boldsymbol{e}}^i\otimes\check{\boldsymbol{e}}^j = \hat{\tau}^i{}_j\,\bar{\boldsymbol{e}}_i\otimes\check{\boldsymbol{e}}^j &&\Leftrightarrow\quad S_8 \equiv R^{\mathrm{t}}\cdot\tau\cdot F.
\end{aligned}
\tag{10}
$$

We map the tensor $\tau$ from the configuration $\hat{B}$ onto the configuration $\bar{B}$:

$$\bar{\tau} \equiv \hat{\tau}^{ij}\,\bar{\boldsymbol{e}}_i\otimes\bar{\boldsymbol{e}}_j = \hat{\tau}_{ij}\,\bar{\boldsymbol{e}}^i\otimes\bar{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\,\bar{\boldsymbol{e}}^i\otimes\bar{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\,\bar{\boldsymbol{e}}_i\otimes\bar{\boldsymbol{e}}^j \Leftrightarrow \bar{\tau}\equiv R^{\mathrm{t}}\cdot\tau\cdot R \quad (\tau = R\cdot\bar{\tau}\cdot R^{\mathrm{t}}). \tag{11}$$

It follows from (7) that the tensors in (9) may be considered as stress tensors obtained by operations of mapping the tensor $\bar{\tau}$ from the configuration $\bar{B}$ onto the configuration $\check{B}$:

$$S_1 = U^{-1}\cdot\bar{\tau}\cdot U^{-1},\quad S_2 = U\cdot\bar{\tau}\cdot U,\quad S_3 = U\cdot\bar{\tau}\cdot U^{-1},\quad S_4 = U^{-1}\cdot\bar{\tau}\cdot U. \tag{12}$$

By analogy, the tensors determined in (10) may be considered as stress tensors obtained by operations of half-mapping of the tensor $\bar{\tau}$ from the configuration $\bar{B}$ onto the configuration $\check{B}$:

$$S_5 = \bar{\tau}\cdot U^{-1},\quad S_6 = U^{-1}\cdot\bar{\tau},\quad S_7 = U\cdot\bar{\tau},\quad S_8 = \bar{\tau}\cdot U. \tag{13}$$

It follows from (11) that the tensor $\bar{\tau}$ is invariant. As is noted above, the tensor $U$ is also invariant. Therefore, it follows from (12) and (13) that the tensors $S_k$ (hereinafter the subscript $k$ runs the values from 1 to 8) are invariant. The tensors $\bar{\tau}$ and $S_k$ form the complete set of invariant stress tensors that can be obtained from the Kirchhoff stress tensor $\tau$ by operations of mapping (including mixed mapping) from the configuration $\hat{B}$ onto the configurations $\check{B}$ and $\bar{B}$. At the same time, these tensors are basis tensors in the family of invariant stress tensors obtained in [5] on the basis of four principles: objectivity, isotropy, conformity, and regularity. These stress tensors have the following names [5]: $S_1$ is the second Piola–Kirchhoff tensor ($S_1 = S_1^{\mathrm{t}}$), $S_2$ is the Green–Rivlin tensor ($S_2 = S_2^{\mathrm{t}}$), $S_3$ and $S_4$ are the first and second Atluri tensors ($S_3 = S_4^{\mathrm{t}}$), $S_5$ and $S_6$ are the first and second Biot tensors ($S_5 = S_6^{\mathrm{t}}$), $S_7$ and

$S_8$ are the first and second Hill tensors ($S_7 = S_8^{\mathrm{t}}$); $\bar{\tau}$ is the Noll tensor or the back-rotated Kirchhoff stress tensor ($\bar{\tau} = \bar{\tau}^{\mathrm{t}}$).

We obtain four stress tensors by operations of mapping of the tensor $\tau$ from the configuration $\hat{B}$ onto the configuration $\tilde{B}$:

$$\begin{aligned} s_1 &\equiv \hat{\tau}^{ij}\, \tilde{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}_j &\Leftrightarrow\quad& s_1 \equiv V^{-1} \cdot \tau \cdot V^{-1}, \\ s_2 &\equiv \hat{\tau}_{ij}\, \tilde{\boldsymbol{e}}^i \otimes \tilde{\boldsymbol{e}}^j &\Leftrightarrow\quad& s_2 \equiv V \cdot \tau \cdot V, \\ s_3 &\equiv \hat{\tau}_i{}^j\, \tilde{\boldsymbol{e}}^i \otimes \tilde{\boldsymbol{e}}_j &\Leftrightarrow\quad& s_3 \equiv V \cdot \tau \cdot V^{-1}, \\ s_4 &\equiv \hat{\tau}^i{}_j\, \tilde{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}^j &\Leftrightarrow\quad& s_4 \equiv V^{-1} \cdot \tau \cdot V. \end{aligned}$$

Using operations of half-mapping of the tensor $\tau$ from the configuration $\hat{B}$ onto the configuration $\tilde{B}$, we obtain four more stress tensors:

$$\begin{aligned} s_5 &\equiv \hat{\tau}^{ij}\, \hat{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}_j = \hat{\tau}_i{}^j\, \hat{\boldsymbol{e}}^i \otimes \tilde{\boldsymbol{e}}_j &\Leftrightarrow\quad& s_5 \equiv \tau \cdot V^{-1}, \\ s_6 &\equiv \hat{\tau}^{ij}\, \tilde{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\, \tilde{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}^j &\Leftrightarrow\quad& s_6 \equiv V^{-1} \cdot \tau, \\ s_7 &\equiv \hat{\tau}_{ij}\, \tilde{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\, \tilde{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}_j &\Leftrightarrow\quad& s_7 \equiv V \cdot \tau, \\ s_8 &\equiv \hat{\tau}_{ij}\, \hat{\boldsymbol{e}}^i \otimes \tilde{\boldsymbol{e}}^j = \hat{\tau}^i{}_j\, \hat{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}^j &\Leftrightarrow\quad& s_8 \equiv \tau \cdot V. \end{aligned}$$

The tensors $s_k$ form the complete set of indifferent stress tensors that can be obtained from the tensor $\tau$ by operations of mapping and half-mapping from the configuration $\hat{B}$ onto the configuration $\tilde{B}$. Following [5], these tensors, including $\tau$, may be called the basis tensors in the family of indifferent stress tensors obtained on the basis of the four principles mentioned above (the objectivity here is understood as indifference instead of invariance).

The tensors $s_k$ and $S_k$ are related via transformations of rotation and back rotation, as $\tau$ and $\bar{\tau}$ [see (11)]: $S_k = R^{\mathrm{t}} \cdot s_k \cdot R$ and $s_k = R \cdot S_k \cdot R^{\mathrm{t}}$. Therefore, the tensors $s_k$ may be called the rotated stress tensors $S_k$. The following equalities are valid: $s_1 = s_1^{\mathrm{t}}$, $s_2 = s_2^{\mathrm{t}}$, $s_3 = s_4^{\mathrm{t}}$, $s_5 = s_6^{\mathrm{t}}$, and $s_7 = s_8^{\mathrm{t}}$. The tensors $s_5$ and $s_6$ may be also called the first and second Bell stress tensors, since the tensor $s_6$ seems to be introduced for the first time in [6] (in [6], the tensor $s_6$ is erroneously considered as symmetric).

Using the operations of half-mapping of the tensor $\tau$ from the actual $\hat{B}$ onto the reference configuration $\check{B}$, we obtain four stress tensors:

$$\begin{aligned} P_1 &\equiv \hat{\tau}^{ij}\, \hat{\boldsymbol{e}}_i \otimes \check{\boldsymbol{e}}_j = \hat{\tau}_i{}^j\, \hat{\boldsymbol{e}}^i \otimes \check{\boldsymbol{e}}_j &\Leftrightarrow\quad& P_1 \equiv \tau \cdot F^{-\mathrm{t}} = R \cdot S_5 = s_5 \cdot R, \\ P_2 &\equiv \hat{\tau}^{ij}\, \check{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\, \check{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}^j &\Leftrightarrow\quad& P_2 \equiv F^{-1} \cdot \tau = S_5 \cdot R^{\mathrm{t}} = R^{\mathrm{t}} \cdot s_5, \\ P_3 &\equiv \hat{\tau}_{ij}\, \check{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\, \check{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}_j &\Leftrightarrow\quad& P_3 \equiv F^{\mathrm{t}} \cdot \tau = S_8 \cdot R^{\mathrm{t}} = R^{\mathrm{t}} \cdot s_8, \\ P_4 &\equiv \hat{\tau}_{ij}\, \hat{\boldsymbol{e}}^i \otimes \check{\boldsymbol{e}}^j = \hat{\tau}^i{}_j\, \hat{\boldsymbol{e}}_i \otimes \check{\boldsymbol{e}}^j &\Leftrightarrow\quad& P_4 \equiv \tau \cdot F = R \cdot S_8 = s_8 \cdot R. \end{aligned}$$

Note that $P_1 = P_2^{\mathrm{t}}$ and $P_3 = P_4^{\mathrm{t}}$. The tensor $P_1$ is called the first Piola–Kirchhoff stress tensor (sometimes, the first Piola–Kirchhoff stress tensor is called the tensor $P_2$ transposed to it). The tensors $P_l$ (the subscript $l$ runs the values from 1 to 4) are non-objective (neither invariant nor indifferent).

Using the operations of half-mapping of the tensor $\tau$ from the actual configuration $\hat{B}$ onto the current back-rotated configuration $\bar{B}$, we obtain two more stress tensors

$$\begin{aligned} T_1 &\equiv \hat{\tau}^{ij}\, \hat{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j = \hat{\tau}_{ij}\, \hat{\boldsymbol{e}}^i \otimes \bar{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\, \hat{\boldsymbol{e}}^i \otimes \bar{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\, \hat{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}^j &\Leftrightarrow\quad& T_1 \equiv \tau \cdot R = R \cdot \bar{\tau}, \\ T_2 &\equiv \hat{\tau}^{ij}\, \bar{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j = \hat{\tau}_{ij}\, \bar{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}^j = \hat{\tau}_i{}^j\, \bar{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}_j = \hat{\tau}^i{}_j\, \bar{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}^j &\Leftrightarrow\quad& T_2 \equiv R^{\mathrm{t}} \cdot \tau = \bar{\tau} \cdot R^{\mathrm{t}}, \end{aligned}$$

for which the equality $T_1 = T_2^{\mathrm{t}}$ is valid. These stress tensors are non-objective.

The stress tensors $\bar{\tau}$, $S_k$, $s_k$, $P_l$, and $T_m$ (the subscript $m$ takes the values 1 and 2) form the complete set of stress tensors obtained from the tensor $\tau$ by operations of mapping, half-mapping, and mixed mapping from the actual configuration $\hat{B}$ onto the configurations $\check{B}$, $\bar{B}$, and $\tilde{B}$. These tensors, including $\tau$, are called the $\tau$-family of natural stress tensors.

Using Eq. (8), we obtain the $s$-family of natural stress tensors: $s$, $\bar{s}$, $J^{-1}S_k$, $J^{-1}s_k$, $J^{-1}P_l$, and $J^{-1}T_m$. The back-rotated Cauchy stress tensor $\bar{s} \equiv R^{\mathrm{t}} \cdot s \cdot R = J^{-1}\bar{\tau}$ [7] is introduced here. The tensor $J^{-1}S_1$ is also called the energetic stress tensor [8]. The families $\tau$ and $s$ compose the class of natural stress tensors.

**Analysis of Natural Stress Tensors.** All the stress tensors introduced have the mechanical meaning of stresses due to the equality of their components to the components of the Kirchhoff $\tau$ or Cauchy $s$ stress tensors in specially chosen basis dyads. It is reasonable to use the objective (invariant or indifferent) stress and strain tensors in constitutive laws. We represent the tensors $\tau$ and $s$ in the spectral form

$$\tau = \tau_i\, \boldsymbol{m}_i \otimes \boldsymbol{m}_i, \quad s = s_i\, \boldsymbol{m}_i \otimes \boldsymbol{m}_i \qquad (\tau_i = J s_i), \tag{14}$$

where $\tau_i$ and $s_i$ are the principal values of the tensors $\tau$ and $s$, respectively, and $\boldsymbol{m}_i$ are the unit vectors of the mutually orthogonal principal axes. The stress state in the material particle is characterized both by the absolute value and by the signs of the principal values of $\tau_i$ or $s_i$. It is desirable to retain the same signs of the principal values of the tensors as those of the tensors $\tau$ or $s$, since the positive principal values of $\tau_i$ or $s_i$ $(J > 0)$ correspond to extension of the material along the corresponding principal axis and the negative values to its compression.

We analyze the invariant stress tensors $\bar{\tau}$, $S_k$, $\bar{s}$, and $J^{-1}S_k$. It follows from (12) and (13) that the Noll stress tensor $\bar{\tau}$ hinges the invariant natural stress tensors of the $\tau$-family. By analogy, the back-rotated Cauchy stress tensor $\bar{s}$ hinges the invariant natural stress tensors of the $s$-family. From Eqs. (11) and (14), we obtain the following spectral representations of the tensors $\bar{\tau}$ and $\bar{s}$: $\bar{\tau} = \tau_i\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i$, $\bar{s} = s_i\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i$, and $\boldsymbol{M}_i \equiv R^{\mathrm{t}} \cdot \boldsymbol{m}_i = \boldsymbol{m}_i \cdot R$. It follows from here that the tensors $\bar{\tau}$ and $\bar{s}$ reproduce exactly the stress state of the material particle in back-rotated principal axes with the orthogonal unit basis vectors $\boldsymbol{M}_i$.

In an arbitrary coordinate system, the components of the second-rank tensors are represented as quadratic matrices of order $3 \times 3$. Equalities (12) and (13) may be treated as matrix transformations of the components of the tensor $\bar{\tau}$ into the components of the tensors $S_k$. Following [5], we use the terminology of matrix transformations of linear algebra for the tensor transformations of Eq. (12) and (13). Tensor transformations in (12) and (13) are equivalence transformations (equivalence transformations retain the rank of matrices). In this sense, all tensors $S_k$ are equivalent to the tensor $\bar{\tau}$. The first two equalities in (12) are congruence transformations (congruence transformations retain both the principal directions and the signs of the principal values). Following [5], we call $S_1$ and $S_2$ the tensors congruent to the tensor $\bar{\tau}$. The second two equalities in (12) are transformations of similarity (transformations of similarity retain the principal values but, in the general case, do not retain orthogonality of the principal directions of the generating symmetric matrix). Following [5], we call the asymmetric tensors $S_3$ and $S_4$ similar to the tensor $\bar{\tau}$. Transformations (13) for matrices are semi-identical. [A semi-identical transformation corresponds to one unit matrix of two (left or right) matrices that border on the generating matrix. In the general case, these transformations retain neither the principal values nor the orthogonality of the principal directions of the generating matrix.] Thus, the asymmetric tensors $S_n$ $(n = \overline{5,8})$ are semi-identical to the tensor $\bar{\tau}$ [5].

Finally, we note that only two congruent stress tensors ($S_1$ and $S_2$) reproduce most exactly the stress state in the material particle among all tensors $S_k$. They retain both the principal directions and the signs of the principal values of the tensor $\bar{\tau}$. Despite the fact that the tensors $S_n$ $(n = \overline{3,8})$ are invariant and equivalent to the tensor $\bar{\tau}$, it is inexpedient to use them in constitutive laws. First, they may distort the mechanical meaning of the stress state of the material particle; second, they are asymmetric in the general case. We consider the symmetric components of these tensors $\tilde{S}_3 \equiv (S_3 + S_4)/2$, $\tilde{S}_5 \equiv (S_5 + S_6)/2$, and $\tilde{S}_7 \equiv (S_7 + S_8)/2$. We call the tensor $\tilde{S}_3$ the symmetric Atluri stress tensor and the tensors $\tilde{S}_5$ and $\tilde{S}_7$ the Jaumann [5] (symmetric Biot) tensor and the Chernykh [4] (symmetric Hill) tensors. Such symmetric tensors have no mechanical meaning of stresses [5]. In the general case, they are not equivalent to the stress tensor $\bar{\tau}$. Note that these stress tensors do not belong to the class of natural stress tensors.

It follows from a similar analysis of indifferent stress tensors that the symmetric stress tensors $s_1$ and $s_2$ are congruent, the asymmetric tensors $s_3$ and $s_4$ are similar, and the tensors $s_n$ $(n = \overline{5,8})$ are semi-identical to the Kirchhoff stress tensor $\tau$.

It is inexpedient to use the asymmetric stress tensors $P_l$ and $T_m$ to formulate constitutive laws, but the tensors $P_1$ or $P_2$ may be used for compact formulation of equations of motion in the reference configuration [3, 8].

The expediency of using the natural stress tensors of the $\tau$-family was considered above. Invariant and indifferent tensors of the $s$-family have the same properties, but non-objective tensors of this family are not used.

Note that all the natural stress tensors are equivalent to the Cauchy stress tensor $s$ or the Kirchhoff stress tensor $\tau$.

**Conclusions.** By means of operations of mapping, half-mapping, and mixed mapping of the Cauchy $s$ and Kirchhoff $\tau$ stress tensors from the actual configuration onto the reference and two intermediate configurations, we obtained some previously known stress tensors and also some new ones, which were called the natural stress tensors. They are divided into two families ($\tau$ and $s$) named by the generating tensors. Each family has four subfamilies. For example, the $\tau$-family has the following subfamilies of stress tensors: invariant $\bar{\tau}$ and $S_k$, indifferent $\tau$ and $s_k$, non-objective $P_l$, and non-objective $T_m$.

The analysis of tensors, which was performed following the technique of [5], showed that the invariant tensors $\bar{\tau}$, $S_1$, and $S_2$ and the indifferent tensors $\tau$, $s_1$, and $s_2$ are optimal for formulation of constitutive laws. In this case, the tensors $S_1$ and $S_2$ are congruent to the tensor $\bar{\tau}$, and $s_1$ and $s_2$ are congruent to the tensor $\tau$. In the $s$-family, the invariant stress tensors $J^{-1}S_1$ and $J^{-1}S_2$ and the indifferent tensors $J^{-1}s_1$ and $J^{-1}s_2$ are congruent to the tensors $\bar{s}$ and $s$, respectively. The expediency of using non-objective stress tensors $P_1$ and $P_2$ in formulation of equations of motion is noted.

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